Section-A

Partial Order Relation

A Relation R on set A is said to be a Partial order relation iff i) Reflexive
ii) Antisymmetric
iii) Transitive

i) Reflexive :- $a R a$, $a \in S$ S = set

ii) Anti Symmetric :- $a R b$ and $b R a$

$\Rightarrow a = b$

iii) Transitive :- $a R b$ and $b R c$

$\Rightarrow a R c$

e.g 1) $A = \{1, 2, 3\}$

$R = \{(1,1)\ (2,2)\ (3,3)\ (1,2)\ (2,3)\ (1,3)\}$

i) Reflexive :- $(1,1)$, $(2,2)$, $(3,3) \in R$

i.e. $a R a$, $a \in A$

$\therefore$ R is Reflexive

ii) Anti Symmetric :- $(1,2) \in R$ but $(2,1) \notin R$

$(2,3) \in R$ but $(3,2) \notin R$

$(1,3) \in R$ but $(3,1) \notin R$

i.e. $a R b$ and $b \not R a$

$a R b$ and $b R a$ iff $a = b$

$\therefore$ R is Antisymmetric

iii) Transitive :- $(1,2) \in R$, $(2,3) \in R$

Also $(1,3) \in R$

i.e. aRb and $bRc \Rightarrow aRc$

$\therefore$ R is Transitive.

$\therefore$ R is Partial order Relation.

eg. 2 Let N be set of Natural Numbers.

$a, b \in N$

Relation R on N : aRb iff a/b
(a Divide b)

Sol: i) Reflexive : $a/a \ \forall a \in N$ [e.g. $2/2$, $5/5$ etc.
$2, 5 \in N$.]

aRa, $a \in N$

$\therefore$ R is Reflexive.

ii) Antisymmetric :- Let aRb and bRa

$\Rightarrow a/b$ and b/a [e.g. $2/4$, $4 \nmid 2$
$2/2$ only]

$\Rightarrow a = b$

R is antisymmetric

iii) Transitive :- Let aRb and bRc

$\Rightarrow a/b$ and b/c [e.g. $2/4$, $4/8$

then a/c $\Rightarrow 2/8$

$\Rightarrow aRc$ $2, 4, 8 \in N$]

$\therefore$ R is Transitive

R is Partial order Relation on N.

2 Partial ordered sets (POSETS)

The set A with partial order relation R is called POSET. Denoted by (A, R) or $(A, \leq)$

e.g 1) Let P(S) set of all subsets of a given set S. S.T. Relation $\subseteq$ is a partial order on P(S) power set.

Sol: i) Reflexive :- We know $A \subseteq A$, $A \in P(S)$

$\therefore \subseteq$ is Reflexive.

ii) Antisymmetric :- Let $A \subseteq B$ and $B \subseteq A$

then $A = B$

[e.g. $A = \{1, 2, 3\}$, $B = \{2, 1, 3\}$ $\Rightarrow A \subseteq B$ and $B \subseteq A \Rightarrow A = B$]

$\therefore \subseteq$ is Antisymmetric

iii) Transitive :- Let $A \subseteq B$ and $B \subseteq C$

then $A \subseteq C$

$\therefore \subseteq$ is Transitive.

[e.g. $A = \{2, 5\}$ $B = \{7, 5, 2, 4\}$, $C = \{1, 4, 6, 2, 5, 7\}$

$A \subseteq B$, $B \subseteq C$ then $A \subseteq C$]

e.g 2 Is the relation Division (/) a partial order on set of all integers (Z)?

Sol: i) Reflexive :- a / a, $a \in Z$ [e.g. $2/2$, $2/-2$]

aRa

$\therefore /$ is Reflexive.

ii) Antisymmetric :- Let aRb and bRa

$\Rightarrow a/b$ and b/a

but $a \neq b$

e.g. $2/-2$ and $-2/2$

but $2 \neq -2$

$\therefore$ / is not Antisymmetric

$\therefore$ / is not POSET on Z.

e.g. 3 Is $(Z^+, <)$ a POSET?

Sol: Reflexive : $a \not< a$, $a \not R a$

e.g. $2 \not< 2$ not Reflexive

$\therefore (Z^+, <)$ is not POSET.

Q 4 Which of are POSETs?

i) $(Z, =)$ ii) $(Z, \neq)$ iii) $(Z, >)$ iv $(Z, \geq)$

Sol i) and iv) are Posets

ii) and iii) are not Posets.

In ii and iii) Reflexive Property not True so it is not Poset.

Homework i) $A = \{1, 2, 3, 4\}$ $R = \{(1,1)\ (1,2)\ (1,3)\ (2,2)\ (3,2),$ $(3,3)\ (4,2)\ (4,3)\ (4,4)\}$ P.O.R?

ii) $R = \{(a,b) \in Z \times Z ;\ |a-b| < 1\}$ on Z P.O.R?

3 **Comparability**: The elements a and b of Poset $(P, \leq)$ is Comparable if $a \leq b$ or $b \leq a$

e.g Ex 1. $(Z^+, |)$, $|$ = Divisible

i) 2 and 4 are Comparable $\because 2 | 4$

ii) 4 and 6 are not Comparable

$\because$ $4 \nmid 6$ also $6 \nmid 4$.

e.g 2. find Two incomparable Elements in

i) $\{P(0,1,2), \subseteq\}$

$\{0\} \nsubseteq \{1\}$, $\{1\} \nsubseteq \{2\}$ is incomparable.

ii) $(\{1,2,4,6,8\}, |)$

4 and 6, 6 and 8 are incomparable

i.e. $4 \nmid 6$ and $6 \nmid 8$

or $6 \nmid 4$ $8 \nmid 6$.

eg 3. find Two Comparable Elements in

$\{P(2,4,5), \subseteq\}$

Sol. $\{2,4\} \subseteq \{2,4,5\}$

are Comparable.

$\{2,5\} \subseteq \{2,5\}$

TOTAL ORDER RELATION:-

A Partial order relation R in set A is called T.O.R if every element $a, b \in A$ s.t. either aRb or bRa

e.g 1. S.T. $A = \{1, 2, 3, 4\}$ under divisibility relation is not Totally ordered.

Sol: $R = \{(1,1), (2,2), (3,3), (4,4), (1,2), (1,3), (1,4), (2,4)\}$

i) Reflexive:- $a|a \ \forall a \in A$ (e.g. $(1,1) \in R$ $1|1$)

aRa

ii) Antisymmetric: Let $\forall a, b \in A$, $a \neq b$ (e.g. $(1,2) \in R$ $1|2$ but $2 \nmid 1$ $(2,1) \notin R$)

$(a, b) \in R$ but $(b, a) \notin R$

$\therefore$ R is Antisymmetric

iii) Transitive:- Let $\forall a, b \in A$ (e.g. $(1,2) \in R$, $(2,4) \in R$ also $(1,4) \in R$ i.e. $1|2$ and $2|4$ also $1|4$)

$(a, b) \in R$, $(b, c) \in R$

then $(a, c) \in R$

$\therefore$ R is Transitive

R is Partial order relation.

iv) Comparability:- $\exists$ 3 & 4 $\in A$

$(3, 4) \notin R$

$\therefore$ R is not Comparable

$\therefore$ R is not TOSET.